

Calculus III: Coursework 1
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Because this sheet was issued later than intended, the deadline will be 5PM on Wednesday Oct 4th. Normal deadlines will be 5PM on Tuesdays. Hand-in is to the green box in the basement of the Maths building.

1. [From the 2005 exam.] Express the following as Fourier series.

(a) $\sin(x + \frac{\pi}{4})$

(b) $\sin^4 x$

It is not necessary to do any integrals. [10]

Answer

We need to realize that any valid sin/cos series answers the stated question.

[3]

(a) $\sin(x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}} \cos x + \frac{1}{\sqrt{2}} \sin x$ [3]

(b) $\sin^4 x = \frac{1}{4}(1 - \cos 2x)^2 = \frac{3}{8} - \frac{1}{2} \cos 2x + \frac{1}{8} \cos 4x$ [4]

2. Suppose we are given

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and

$$\frac{df}{dx} = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx).$$

Derive A_0, A_n, B_n in terms of a_0, a_n, b_n with brief explanations. [10]

Answer

Differentiate the first line and compare coefficient with the second line, which gives

$$A_0 = 0, \quad A_n = nb_n, \quad B_n = -na_n.$$

Marks: Attempting to differentiate term by term (5), correct explanations (5).

3. Show that for $-\pi < x < \pi$

$$x = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx. \quad (1)$$

Now, with the help of (1) derive

$$1 - \frac{1}{3} + \frac{1}{5} + \dots = \frac{\pi}{4}.$$

[15]

Answer

We need the Fourier series for x , and since it is odd we only need the sine series with

$$b_n = \frac{2}{\pi} \int_0^\pi x \sin nx \, dx$$

Integrating by parts we get

$$b_n = -\frac{2 \cos n\pi}{n} = \frac{2(-1)^{n+1}}{n}$$

which gives equation (1).

For the series, put $x = \pi/2$ in (1).

Marks: Correct formula for b_n (5), integration and simplification (5), putting $x = \pi/2$ and simplifying (5).

4. Write down the Fourier series for

$$f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ x & \text{if } 0 < x < \pi. \end{cases}$$

[Hint: use the previous answer and the result in lectures for $|x|$.] Simplify the series at $x = \pi$ to derive a series for π^2 . [20]

Answer

The given $f(x)$ is

$$\frac{x + |x|}{2}$$

which is just the mean of the example in lectures and the example above. Taking the mean of their Fourier series, we have

$$f(x) = \frac{\pi}{4} + \sum_n \frac{(-1)^{n+1}}{n} \sin nx - \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos nx$$

At $x = \pi$ the series equals the mean of the discontinuous values. That is

$$\frac{\pi}{2} = \frac{\pi}{4} - \frac{2}{\pi} \sum_{n \text{ odd}} \frac{\cos n\pi}{n^2}$$

and hence

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

Marks Writing down correct series (5), remembering about the continuity (5), simplifying to the correct identity (10). (The added hint this year made the question easier and probably 5 marks should have been transferred to the next question.)

5. Find the Fourier cosine series for $f(x) = x^2$ on $(0, \pi)$. [10]

Answer

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi x^2 \cos nx dx \\ &= \frac{2}{\pi} \left(\left[\frac{x^2 \sin nx}{n} \right]_0^\pi - \int_0^\pi \frac{2x \sin nx}{n} dx \right) \quad (n \neq 0) \\ &= \frac{2}{\pi} \left(\left[\frac{2x \cos nx}{n^2} \right]_0^\pi - \int_0^\pi \frac{2 \cos nx}{n^2} dx \right) \\ &= \frac{2}{\pi} \left(\frac{2\pi(-1)^n}{n^2} - \left[\frac{2 \sin nx}{n^3} \right]_0^\pi \right) \\ &= \frac{4(-1)^n}{n^2} \end{aligned}$$

and if $n = 0$, integration gives $a_0 = [2x^3/3]_0^\pi/\pi = 2\pi^2/3$, so series is

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos nx}{n^2}$$

Marks a_n 6, a_0 2, putting it together 2.

6. We are given an odd function $f(x)$ and we want to approximate it in the domain $[-\pi, \pi]$ by a finite series

$$\sum_{n=1}^N s_n \sin nx$$

with suitable coefficients s_n . To find the optimal s_n we first define a function which measures the ‘badness’ of the approximation

$$R = \int_0^\pi \left(f(x) - \sum_{n=1}^N s_n \sin nx \right)^2 dx$$

and then minimize R by requiring

$$\frac{\partial R}{\partial s_m} = 0.$$

Solve this last equation for s_m and compare with the usual Fourier sine coefficients b_m . [25]

Answer

For the derivative we get

$$2 \int_0^\pi \sin mx \left(f(x) - \sum_{n=1}^N s_n \sin nx \right) dx = 0$$

or

$$\int_0^\pi f(x) \sin mx \, dx = \sum_{n=1}^N \int_0^\pi \sin mx \sin nx \, dx = \frac{\pi}{2} s_m .$$

Thus the s_m coefficients are the same as the Fourier sine coefficients.

Marks: Differentiating correctly (10), attempting to use orthogonality (5), simplifying correctly (5), relating to b_n (5).

7. Use the Fourier series from question 3 and Parseval's relation [which recall is

$$\frac{1}{\pi} \int_{-\pi}^{\pi} f^2(x) \, dx = \frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

in the usual notation] to derive an infinite series for π^2 . [10]

Answer

Substituting in Parseval's relation we have

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

which simplifies to

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

Marks half-sensible use of Parseval's relation (5), correct simplification (5).

Note The relation between the result here and in question 4 arises because

$$\sum_{n \text{ even}} \frac{1}{n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}, \text{ so } \sum_{n \text{ odd}} \frac{1}{n^2} = \frac{3}{4} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

and $\frac{3}{4} \frac{\pi^2}{6} = \frac{\pi^2}{8}$.